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CON-K- NORMAL AND UNITARY POLYNOMIAL MATRICES

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ABSTRACT

In this paper we introduce con-k-normal polynomial and con- k-unitary polynomial matrices and study some of its properties.

KEYWORDS:

Con-k-normal polynomial matrix, con-k-unitary polynomial matrix.

I.INTRODUCTION

Let $C^{n \times n}$ be set of all polynomial matrices over the complex field of order n. Let ‘ k’ be a fixed product of disjoint transpositions in $S_n = \{1,2,3\ldots,n\}$ and K be the associated permutation matrix and $K^2 = I$, $K = K^T = K^*$. Any matrix A is said to be con-k-normal if $AA^*K = \overline{KA^*A} = KA^T\bar{A}$ [2], where $\bar{A}$, A^T, A^* denote conjugate, transpose, conjugate transpose of a matrix A respectively. In this paper, we study the concept of con-k-normal and unitary polynomial matrices as a generalization of k-normal and unitary polynomial matrices[1].

II. CON-K-NORMAL POLYNOMIAL MATRICES

In this section, we give the definition of the con-k-normal polynomial matrix and some of its basic algebraic properties are studied, as a extension of k-normal polynomial matrices[1].

Definition: 2.1

A con-k-normal polynomial matrix is a polynomial matrix whose coefficient matrices are con-k-normal matrix.

Example: 2.2

$A(x) = \begin{bmatrix} ix+1 & 1 \\ 1 & ix+i \end{bmatrix}$ is a con-k-polynomial matrix.

Here, $A(x) = A_1 x + A_0$

$= \begin{bmatrix} i & 0 \\ 1 & i \end{bmatrix} x + \begin{bmatrix} i & 1 \\ 0 & i \end{bmatrix}$, where A_0, A_1 are con-k-normal matrices.

Theorem: 2.3

If $A(\lambda)$, $B(\lambda) \in C^{n \times n}$ are con-k-normal polynomial matrices and $A(\lambda)B(\lambda) = B(\lambda)A(\lambda)$, then $A(\lambda)B(\lambda)$ is a con-k-normal polynomial matrix.

Proof :

Let $A(\lambda) = A_0 + A_1\lambda + \ldots + A_n\lambda^n$ and $B(\lambda) = B_0 + B_1\lambda + \ldots + B_n\lambda^n$ be con-k-normal polynomial matrices, A_0 , $A_1 \ldots A_n$ and B_0 , $B_1 \ldots B_n$ are con-k-normal matrices. Also given ,

$A(\lambda)B(\lambda) = B(\lambda)A(\lambda)$

$A(\lambda)B(\lambda) = A_0 B_0 + (A_0B_1+A_1B_0)\lambda+\ldots+(A_0B_n+A_1B_{n-1}+\ldots+A_nB_0)\lambda^n$

$B(\lambda)A(\lambda) = B_0A_0 + (B_0A_1+B_1A_0)\lambda+\ldots+(B_0A_n+B_1A_{n-1}+\ldots+B_nA_0)\lambda^n$

Here each coefficient of λ and constants terms are equal.

(i.e) $A_0B_0 = B_0A_0$

$A_0B_1 + A_1B_0 = B_0A_1+B_1A_0$

$\Rightarrow$ $A_0B_1 = B_0A_1$ and $A_1B_0 = B_1A_0$ ....

$\Rightarrow$ $A_nB_0 = B_0A_n$, $A_1B_{n-1} = B_1A_{n-1}$,..., $A_0B_n = B_nA_0$

Now we have to prove $A(\lambda)B(\lambda)$ is con-k-normal polynomial matrix.

$A(\lambda)\, B(\lambda)\, [A(\lambda)\, B(\lambda)]^* K = A(\lambda)\, B(\lambda)\, A^*(\lambda)B^*(\lambda)K$

$= A(\lambda)\, A^*(\lambda)\, B(\lambda)\, B^*(\lambda)\, K \quad = A(\lambda)\, A^*(\lambda)\, K\, B^T(\lambda)\, \overline{B(\lambda)}$

$= K\, A^T(\lambda)\, \overline{A(\lambda)}\, B^T(\lambda)\overline{B(\lambda)} \quad = KA^T(\lambda)\, B^T(\lambda)\, \overline{A(\lambda)}\, \overline{B(\lambda)}$

$= K[B(\lambda)\, A(\lambda)]^T\, \overline{[B(\lambda)\, A(\lambda)]} \;\; = K\, [A(\lambda)\, B(\lambda)]^T\, \overline{[A(\lambda)\, B(\lambda)]}$.

Hence $A(\lambda)\, B(\lambda)$ is con-k-normal polynomial matrix.

Theorem:2.4

Let $A(\lambda) \in C^{n \times n}$, then the following conditions are equivalent:

(i) $A(\lambda)$ is con-k-normal polynomial matrix.
(ii) $\overline{A(\lambda)}$ is con- k-normal polynomial matrix.
(iii) $A^T(\lambda)$ is con-k-normal polynomial matrix.
(iv) $A^*(\lambda)$ is con-k-normal polynomial matrix.

(v) $hA(\lambda)$ is con-k-normal polynomial matrix where h is a real number.

Proof:

(i) <=> (ii):

A(λ) is con-k-normal polynomial $\Leftrightarrow A(\lambda)\, A^*(\lambda)\, K = K\, A^T(\lambda)\, \overline{A(\lambda)}$

$\Leftrightarrow \overline{A(\lambda)\, A^*(\lambda)\, K} = \overline{K\, A^T(\lambda) A(\lambda)}$

$\Leftrightarrow \overline{A(\lambda)}\, A^T(\lambda) K = K\, A^*(\lambda)\, A(\lambda).$

$\Leftrightarrow \overline{A(\lambda)}$ is con- k-normal polynomial matrix.

(i) <=> (iii):

A(λ) is con-k-normal polynomial $\Leftrightarrow A(\lambda)A^*(\lambda)\, K = KA^T(\lambda)\, \overline{A(\lambda)}$

$\Leftrightarrow (A(\lambda)\, A^*(\lambda)\, K\,)^T = (\, K\, A^T(\lambda)\, \overline{A(\lambda)}\,)^T$

$\Leftrightarrow K^T\, (A^*(\lambda))^T\, A^T(\lambda) = \overline{A^T(\lambda)}\, (A^T(\lambda)\,)^T\, K^T$

$\Leftrightarrow K\, \overline{A(\lambda)}\, A^T(\lambda) = A^*(\lambda)\, A(\lambda)\, K$

Pre and post multiply by K on both sides,

$\Leftrightarrow \overline{A(\lambda)}\, A^T(\lambda)\, K = K\, A^*(\lambda)\, A(\lambda)$

$\Leftrightarrow A^T(\lambda)$ is con-k-normal polynomial matrix.

(i) <=> (iv):

A(λ) is con-k-normal polynomial $\Leftrightarrow A(\lambda)A^*(\lambda)\, K = KA^T(\lambda)\, \overline{A(\lambda)}$

$\Leftrightarrow (A(\lambda)A^*(\lambda)K)^* = (KA^T(\lambda)\, \overline{A(\lambda)}\,)^*$

$\Leftrightarrow K^*\, (A^*(\lambda)\,)^*\, A^*(\lambda) = (\overline{A(\lambda)}\,)^*\, (A^T(\lambda))^*\, K^*$

$\Leftrightarrow K\, A(\lambda)\, A^*(\lambda) = A^T(\lambda)\, \overline{A(\lambda)}\, K$

Pre and post multiply by K on both sides,

$\Leftrightarrow A(\lambda)A^*(\lambda)\, K = KA^T(\lambda)\, \overline{A(\lambda)}$

$\Leftrightarrow A^*(\lambda)$ is con-k-normal polynomial matrix.

(i) <=> (v):

A(λ) is con-k-normal polynomial $\Leftrightarrow A(\lambda)A^*(\lambda)\, K = KA^T(\lambda)\, \overline{A(\lambda)}$

$\Leftrightarrow h^2(\, A(\lambda)A^*(\lambda)\, K) = h^2(\, KA^T(\lambda)\, \overline{A(\lambda)}\,)$

$\Leftrightarrow (\, h\, A(\lambda))\, (\, h\, A^*(\lambda)\,)K = K(\, h\, A^T(\lambda)\,)(\, h\, \overline{A(\lambda)}\,)$

$\Leftrightarrow (\, h\, A(\lambda))\, (\, h\, A^*(\lambda)\,)K = K\, (\, h\, A^T(\lambda)\,)(\, \overline{hA(\lambda)}\,)$

$\Leftrightarrow hA(\lambda)$ is con-k-normal polynomial matrix.

Theorem: 2.5

If $A(\lambda), B(\lambda) \in C^{n\times n}$ are con- k-normal polynomial matrices $A(\lambda)\, B^{*}(\lambda)K = K\, B^{T}(\lambda)\, \overline{A(\lambda)}$ and $B(\lambda)\, A^{*}(\lambda)K = K\, A^{T}(\lambda)\, \overline{B(\lambda)}$, then $A(\lambda) + B(\lambda)$ and $A(\lambda) - B(\lambda)$ are con-k-normal polynomial matrix.

Proof:

Since $A(\lambda)$ and $B(\lambda)$ are con-k-normal polynomial matrices.

We have $A(\lambda)A^{*}(\lambda)K = KA^{T}(\lambda)\overline{A(\lambda)}$ -------(1)

and $B(\lambda)B^{*}(\lambda)K = KB^{T}(\lambda)\, \overline{B(\lambda)}$ -------(2)

$$
\begin{aligned}
A(\lambda) + B(\lambda))(A(\lambda)+B(\lambda))^{*}K &= A(\lambda)\, A^{*}(\lambda)K + A(\lambda)\, B^{*}(\lambda)K + B(\lambda)\, A^{*}(\lambda)K + B(\lambda)B(\lambda)^{*}K \\
&= KA^{T}(\lambda)\, \overline{A(\lambda)} + K\, B^{T}(\lambda)\, \overline{A(\lambda)} + K\, A^{T}(\lambda)\, \overline{B(\lambda)} + KB^{T}(\lambda)\, \overline{B(\lambda)} \\
&= KA^{T}(\lambda)\, (\overline{A(\lambda)} + \overline{B(\lambda)}) + K\, B^{T}(\lambda)\, (\overline{A(\lambda)} + \overline{B(\lambda)}) \\
&= K(A^{T}(\lambda) + B^{T}(\lambda))(\overline{A(\lambda)} + \overline{B(\lambda)}) \\
&= K(A(\lambda) + B(\lambda))^{T}\, \overline{(A(\lambda) + B(\lambda)}
\end{aligned}
$$

Hence $A(\lambda) + B(\lambda)$ is con-k-normal polynomial matrix.
Similarly we can prove, $A(\lambda) - B(\lambda)$ is con-k-normal polynomial matrix.

Theorem: 2.6

If $A(\lambda) \in C^{n\times n}$ is con-k-normal polynomial matrix then $iA(\lambda)$, $-iA(\lambda)$ are con-k-normal polynomial matrix.

Proof:

$A(\lambda)$ is con-k-normal polynomial => $A(\lambda)A^{*}(\lambda)K = KA^{T}(\lambda)\overline{A(\lambda)}$

$\Leftrightarrow i^{2}(A(\lambda)A^{*}(\lambda)K) = i^{2}(KA^{T}(\lambda)\, \overline{A(\lambda)})$

$\Leftrightarrow (i\, A(\lambda))\, (-(i\, A(\lambda))^{*})K = K\, (i\, A^{T}(\lambda))(-(i\overline{A(\lambda)})$

$\Leftrightarrow (i\, A(\lambda))\ (i\, A(\lambda))^{*}K = K\, (iA(\lambda))^{T}\ (\overline{iA(\lambda)})$

$\Leftrightarrow$ $iA(\lambda)$ is con-k-normal polynomial matrix.

Similarly,

we can prove $-iA(\lambda)$ is con-k-normal polynomial matrix.

III. CON-K-UNITARY POLYNOMIAL MATRICES

In this section we have given the definition of con-k-unitary polynomial matrices and obtained its equivalent conditions.

Definition: 3.1

A con-k-unitary polynomial matrix is a polynomial matrix whose coefficient matrices are con-k-unitary matrices.

Example: 3.2

$$A(x) = \begin{bmatrix} \frac{i}{\sqrt{2}}x + i & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & i \end{bmatrix}$$ is a con-k-unitary polynomial matrix.

Here, $A(x) = A_1 x + A_0$

$$= \begin{bmatrix} \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{bmatrix} x + \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix},$$ where A_0, A_1 are con-k-normal matrices.

Theorem: 3.3

Let $A(\lambda) \in C^{n \times n}$, then the following conditions are equivalent

(i) $A(\lambda)$ is con-k-unitary polynomial matrix.
(ii) $\overline{A(\lambda)}$ is con- k-unitary polynomial matrix.
(iii) $A^T(\lambda)$ is con-k-unitary polynomial matrix.
(iv) $A^*(\lambda)$ is con-k-unitary polynomial matrix.
(v) $hA(\lambda)$ is con-k-unitary polynomial matrix, where h is a real number.

Proof:

The proof is similar to that of theorem 2.4.

Theorem: 3.4

If $A(\lambda)$, $B(\lambda) \in C^{n \times n}$ are con-k-unitary polynomial matrices, then $A(\lambda)B(\lambda)$ is con-k-unitary polynomial matrices.

Proof:

$A(\lambda)$ is k- unitary polynomial then $A(\lambda)A^*(\lambda) K = KA^T(\lambda)\overline{A(\lambda)} = K$.

$B(\lambda)$ is k- unitary polynomial then $B(\lambda)B^*(\lambda)K = KB^T(\lambda)\overline{B(\lambda)} = K$.

$$\begin{aligned}(A(\lambda)B(\lambda))(A(\lambda)B(\lambda))^*K &= A(\lambda)B(\lambda)A^*(\lambda)B^*(\lambda)K \\ &= A(\lambda)A^*(\lambda)B(\lambda)B^*(\lambda)K \\ &= A(\lambda)A^*(\lambda)K = K = KA^T(\lambda)\overline{A(\lambda)}B^T(\lambda)\overline{B(\lambda)} \\ &= KA^T(\lambda)B^T(\lambda)\overline{A(\lambda)B(\lambda)} \\ &= K(A(\lambda)B(\lambda))^T(\overline{A(\lambda)B(\lambda)})\end{aligned}$$

Hence $(A(\lambda)B(\lambda))(A(\lambda)B(\lambda))^*K = K(A(\lambda)B(\lambda))^T\overline{A(\lambda)B(\lambda)} = K$.

$A(\lambda)B(\lambda)$ is con-k-unitary polynomial matrix.

Corollary: 3.5

If $A(\lambda)$, $B(\lambda) \in C^{n \times n}$ are con-k-unitary polynomial matrices, then $B(\lambda)A(\lambda)$ is con-k-unitary polynomial matrices.

Theorem: 3.6

Let $A(\lambda), B(\lambda) \in C^{n\times n}$. If $A(\lambda)$ and $B(\lambda)$ are con-k-unitary polynomial matrices, then $B(\lambda)A(\lambda)$ is con-unitary polynomial matrices.

Proof:

Since $A(\lambda)$ and $B(\lambda)$ is k- unitary polynomial then $A(\lambda)A^*(\lambda) K = KA^T(\lambda)\overline{A(\lambda)} = K$ and $B(\lambda)B^*(\lambda)K = KB^T(\lambda) \overline{B(\lambda)} = K$.

From the above two equations, we have

$A(\lambda)A^*(\lambda) K B(\lambda)B^*(\lambda)K = KA^T(\lambda) \overline{A(\lambda)} KB^T(\lambda)\overline{B(\lambda)} = I_n$

$\Rightarrow K B(\lambda)B^*(\lambda)K = K A^T(\lambda) \overline{A(\lambda)} K = I_n$

$\Rightarrow B(\lambda)B^*(\lambda) = A^T(\lambda)\overline{A(\lambda)} = I_n.$

$\Rightarrow B(\lambda)K^2 B^*(\lambda) = A^T(\lambda) K^2\overline{A(\lambda)} = I_n$

$\Rightarrow B(\lambda)A(\lambda)A^*(\lambda)KK B^*(\lambda) = A^T(\lambda) KK B^T(\lambda) \overline{B(\lambda)}\ \overline{A(\lambda)} = I_n.$

$\Rightarrow B(\lambda)A(\lambda)(B(\lambda)A(\lambda))^* = (B(\lambda)A(\lambda))^T \overline{B(\lambda)}\ \overline{A(\lambda)} = I_n.$

=> $B(\lambda)A(\lambda)$ is con-unitary polynomial matrix. Hence the proof.

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