EXACT SOLUTIONS FOR SOME NONLINEAR FRACTIONAL PARABOLIC EQUATIONS

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ABSTRACT

In this work, we have generalized the nonlinear parabolic equations: the Burger's equation, the Fitzhugh Nagaimo equation and the general nonlinear parabolic equation, which was solved by Wazwaz, i.e., we solved in a case space-time fractional derivative (1-3) by using the tanh-coth method.

Keywords: *Nonlinear space - time fractional (PDEs), tanh-coth method, exact solutions, Taylor series of first order approximation of non differentiable functions.*

1. INTRODUCTION

Importance of fractional differential equations in studies some natural phenomena, has spurred many researchers for the study and discusses some of the well-knownclassicaldifferentialequations, (see e.g. [11-25]), by replacing some its derivatives or all by fractional derivatives. In this paper we have considered the equations:

(I) The space time fractional Burger's equation

$\frac{\partial^{\alpha}u}{\partial t^{\alpha}} = \frac{\partial^{2\beta}u}{\partial x^{2\beta}} + au\frac{\partial^{\beta}u}{\partial x^{\beta}}, \quad 0 < \alpha, \beta < 1$(1) (II) The space time fractional Fitzhugh Nagumo equation

$\frac{\partial^{\alpha}u}{\partial t^{\alpha}} = \frac{\partial^{2\beta}u}{\partial x^{2\beta}} - u(1 - u(a - u), \quad 0 < \alpha, \beta < 1$(2)(III)The general nonlinear space time fractional parabolic equation

$\frac{\partial^{\alpha}u}{\partial t^{\alpha}} = \frac{\partial^{2\beta}u}{\partial x^{2\beta}} + au + bu^{n}, \quad 0 < \alpha, \beta < 1.$(3) By using tanh-coth method. These equations discussed by wazwaz[1] when $\alpha = \beta = 1$.This paper is arranged as follows: In Section 2, we present concepts that make the chain rule is valid for fractional derivatives. In Section 3, we give the description for main steps of the tanh-coth method. In Section 4, we apply this method to finding exact solutions for the space-time fractional equations which we have stated above.

2. PRELIMINARIES

In this section we used the definition of fractional derivative via difference derivative and the Generalized Handmaid'stheorem for finding the Taylor series of first order approximation of the non-differentiable functions and using the latter for concludepower rule and the chain rule of non-differentiable functions, and we used these rules with Eq. (21)to get the Eq.(22) and using E.g. (22) to convert the FPDE (20)into the (ODE) (23).

2.1 Fractional derivative via fractional difference

Definition **(2.1.1**) $f: \mathbb{R} \to \mathbb{R}$, denote continuous (but not differentiable function) and let $h > 0$ denote a constant discretization span. Define the forward operator [2].

$FW(h)f(x) = f(x+h)$(4)Then the fractional difference of order $\alpha \in \mathbb{R}$, $0 < \alpha \leq 1$ of $f(x)$ is defined by expression

$\Delta^{\alpha} f(x) = (\, FW - 1)^{\alpha} = \sum_{k=0}^{\infty}(-1)^k \binom{\alpha}{k} f[x+(\alpha-k)h]$(5)And its fractional derivative of order α is

$f^{(\alpha)}(x) = \lim_{h\to 0} \frac{\Delta^{\alpha} f(x)}{h^{\alpha}}$(6)And from this definition we can derive the alternative

$f^{(\alpha)}(x) = \frac{1}{\Gamma(-\alpha)} \int_0^x (x-u)^{-\alpha-1}\big(f(u)-f(0)\big)du, \alpha < 0$(7)Forpositive α, one will set

$f^{(\alpha)}(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x (x-u)^{-\alpha}\big(f(u)-f(0)\big)du, 0 < \alpha < 1$(8)And

$f^{(\alpha)}(x) = \frac{1}{\Gamma(1-\alpha+n)} \frac{d^n}{dx^n} \int_0^x (x-u)^{-\alpha+n}\big(f(u)-f(0)\big)du, n < \alpha < n+1$(9)

2.2. Generalized Hadamard's Theorem

We denote by$f(x) \in C^{m\alpha}(U)$the space of functions $f(x)$which, are continuously m timesαth-differentiable. **Hadamard's. Theorem Generalized.** Any function $f(x) \in C^{\alpha}(U)$in a neighborhood of a point $x_0$can be decomposed in the form [3].

$f(x) = f(x_{0)} + \frac{(x-x_0)^{\alpha}}{\alpha!} g(x_0)$(10)

Where$g(x) \in C^{m\alpha}(U)$

If we use this theorem to $g(x)$ in Eq. (10) again we get

$f(x) = f(x_0) + \frac{(x-x_0)^{\alpha}}{\alpha!} g_1(x_0) + \frac{(x-x_0)^{\alpha}}{(\alpha!)^2} g_2(x_0)$(11)**2.3. Application to Fractional Taylor Series of First Order**

Corollary (2.3.2).As a result of the generalized Hadamard's theorem, one has as well Taylor series of first order approximation [3].

$f(x) = f(x_0) + \frac{(x-x_0)^{\alpha}}{\alpha!} f^{\alpha}(x) + O(h)^{2\alpha}$(12)Note that from proof of this Corollary

$\Delta^{\alpha}f(x) = \alpha!\Delta f(x) - O(h)^{2\alpha}$(13)Whereby we obtain

$\Delta^{\alpha}f(x) \cong \Gamma(1+\alpha)\Delta f(x)$(14)Or in a differential form

$d^{\alpha}f(x) \cong \Gamma(1+\alpha)df(x)$(15)We note that from (13)

$f^{(\alpha)}(x) = \lim_{h\to 0}\frac{\Delta^{\alpha}f(x)}{h^{\alpha}} = \Gamma(1+\alpha)\lim_{h\to 0}\frac{\Delta^{\alpha}f(x)}{h^{\alpha}}$(16)**Corollary (2.3.2).**The following equalities hold, which are [5]

$D^{\alpha}x^{\beta} = \Gamma^{-1}(1+\beta)\Gamma(\beta-\alpha+1)x^{\beta-\alpha}, \quad \beta>0(17)f^{\alpha}[u(x))] = f_u^{(\alpha)}(u)(u_x')^{\alpha}$(18)

$= f_u'(u)u^{(\alpha)}(x)$(19)Where f in Eq. (18) is non-differentiable w.r.t u, while u is differentiable w.r.t x, f in Eq. (19) is differentiable w.r.t u, while u is non-differentiablew.r.t x.

Proof: Proof (17): From Eq. (12) let $x-x_0=h$, we have

$$D^{\alpha}x^{\beta} = \Gamma(1+\alpha)\frac{(x_0+h)^{\beta}-x_0^{\beta}}{h^{\alpha}} - O(h)^{2\alpha}$$

$$= \Gamma(1+\alpha)h^{-\alpha}\left(\sum_{k=0}^{\beta}\frac{\Gamma(1+\beta)}{\Gamma(k+1)\Gamma(\beta-k+1)}h^k x_0^{\beta-k} - x_0^{\beta}\right) - O(h)^{2\alpha}$$

$$= \Gamma(1+\alpha)\left(\sum_{k=0}^{\beta}\frac{\Gamma(1+\beta)}{\Gamma(k+1)\Gamma(\beta-k+1)}h^{k-\alpha}x_0^{\beta-k}\right) - O(h)^{2\alpha}$$

And by making h tend to zero we obtain

$$\begin{cases} 0, & k>\alpha \\ \Gamma^{-1}(1+\beta)\Gamma(\beta-\alpha+1)x^{\beta-\alpha}, & k=\alpha \\ \infty, & k<\alpha \end{cases}$$

Proof (19): we have from Eq. (13)

$$\Delta^{\alpha}f(x) = \alpha!\Delta f(x) - O(h)^{2\alpha}$$

This provides, for small h,

$$h^{-\alpha}\Delta^{\alpha}f(x) = \alpha!\,h^{-\alpha}\Delta f(x) - h^{-\alpha}O(h)^{2\alpha}$$

And by making h tend to zero we obtain

$$\frac{d^{\alpha}f(u)}{dx^{\alpha}} = \frac{\alpha!df}{dx^{\alpha}} = \frac{df}{du}\frac{\alpha!du}{dx^{\alpha}} = \frac{df}{du}\frac{d^{\alpha}u}{dx^{\alpha}}.$$

3. OUTLINE OF THE TANH--COTH METHOD

In this section we gave a brief description for the main steps of the tanh-coth method. For that, consider a space-time fractional nonlinear parabolic equation in two independent variables x, t and a dependent variable u

$P\left(u, D_t^{\alpha}u, D_x^{\beta}u, D_x^{2\beta}u, D_x^{3\beta}u, \ldots\right) = 0, \quad 0 < \alpha, \beta < 1$(20) **Step1**.We use the transformation:

$u(x,t) = u(\xi), \ \ \xi = \frac{kx^{\beta}}{\Gamma(1+\beta)} - \frac{ct^{\alpha}}{\Gamma(1+\alpha)}$(21) Where c andk are arbitrary constants different from zero. Based on this and using Eq. (17) and Eq. (19) we can easily drive:

$$\frac{\partial^{\alpha}}{\partial t^{\alpha}} = -c\frac{d}{d\xi}$$

$\frac{\partial^{\beta}}{\partial t^{\beta}} = k\frac{d}{d\xi}\frac{\partial^{2\beta}}{\partial t^{2\beta}} = k^2\frac{d}{d\xi}$(22)And so on. Eq. (22) changes the Eq. (20) to an (ODE) as:

$Q(u, u', u'', u''', \ldots) = 0$(23) Where Q is a polynomial of u and its derivatives and the superscripts indicate the ordinary derivatives with respect to ξ. If possible, we should integrate Eq. (23) term by term one or more times.

Step2. Suppose the solutions of Eq. (23) can be expressed as a polynomial of Y in the form

$u(\xi) = S(Y) = \sum_{i=-M}^{M} a_i Y$(24) Where a_i(i = 0, 1... M) (M is positive number, called the balance number) are constants to be determined later, while the function Y= tanh (μξ), Y satisfies the differential equation

$$\frac{dY}{d\xi} = \mu(1 - Y^2)$$

So by using chain rule we can write:

$$\frac{d}{d\xi} = \frac{dY}{d\xi}\frac{d}{dY} = \mu(1 - Y^2)\frac{d}{dY}$$

$$\frac{d^2}{d\xi^2} = \frac{d}{d\xi}\left(\frac{dY}{d\xi}\frac{d}{dY}\right)$$

$$= \left(\frac{dY}{d\xi}\right)\left(\frac{dD_Y}{d\xi}\right) + \left(\frac{d}{dY}\right)\left(\frac{d^2Y}{d\xi^2}\right) = \left(\frac{dY}{d\xi}\right)^2\left(\frac{d^2}{dY^2}\right) + \left(\frac{d}{dY}\right)\left(\frac{d^2Y}{d\xi^2}\right)$$

$= -2Y\mu^2(1-Y^2)\frac{d}{dY} + \mu^2(1-Y^2)^2\left(\frac{d^2}{dY^2}\right)$(25)And so on, where $D_Y = \frac{d}{dY}$, μ is a constant. The positive integer M in Eq.(24)can be determined by considering the homogeneous balance between the highest-order derivatives and nonlinear terms appearing in Eq.(23) If M is equal to a fractional or negative number, we can take the following transformations [4].

1- When $M = \frac{q}{p}$(where M=$\frac{q}{p}$) is a fraction in lowest terms), we let

$u(\xi) = v^{\frac{q}{p}}(\xi)$(26) Substituting Eq.(26) into Eq.(23) and then determine the value of M in new Eq.(23)

2- When M is a negative integer, we let

$u(\xi) = v^M(\xi)$(27) Substituting Eq.(27) into Eq.(23) and return to determine the value of M once again.

Step3. Substituting from Eq. (25) into the Eq. (23) we get

$R(Y, S(Y), S'(Y), S''(Y), \ldots) = 0$(28)

Step4. Substituting Eq. (24) into the Eq. (28) yields an equation in powers of Y. We then collect all coefficients of powers of Y in the resulting equation where these coefficients have to vanish. This will give a system of algebraic involving the parameters a_k,(k= 0, 1, 2…M), μ, c and having determined these parameters we obtain an analytic solution u(x, t) in a closed form.

4. APPLICATIONS

1. The space-time fractional Burger's equation

$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^{2\beta} u}{\partial x^{2\beta}} + au\frac{\partial^\beta u}{\partial x^\beta}$, $0 < \alpha, \beta < 1$(29)Substituting from Eq. (22) changes the FPDE (29) into the following nonlinear (ODE)

$cu' + k^2u' + akuu' = 0$(30)Integrating Eq. (30) with respect to ξ and setting the integration constant to zero, we get

$cu + k^2u' + \frac{ak}{2}u^2 = 0$(31)Balancing u′ with u² we obtain M=1.Thus Eq. (24) becomes

$u(\xi) = S(Y) = a_{-1}Y^{-1} + a_0 + a_1Y$(32)Substituting from Eq. (25) into Eq. (31) we get

$cS + \mu k^2(1-Y^2)\frac{dS}{dY} + \frac{ak}{2}S^2 = 0$(33)

Substituting Eq. (32) into Eq. (33) then by using maple package we get a system of algebraic equations for a_{-1}, a, a_1 and μ, c in the form:

$Y^{-2}: \mu k^2 a_{-1} - \frac{1}{2} a k a_{-1}^2 = 0$

$Y^{-1}: a k a_0 a_{-1} + c a_{-1} = 0$

$Y^0: \mu k^2 a_1 + \mu k^2 a_{-1} + \frac{1}{2} k a a_0^2 + c a_0 + a k a_1 a_{-1} = 0$

$Y: c a_1 + a k a_0 a_1 = 0$

$Y^2: \mu k^2 a_1 - \frac{1}{2} a k a_1^2 = 0$

Solving these resulting equations using Maple, we obtain the following three sets of solutions:

1. $a_{-1} = 0,\ a_0 = \frac{-c}{ak}, a_1 = \frac{\mp c}{ak}, \mu = \frac{\mp c}{2k^2}$
2. $a_{-1} = \frac{\mp c}{ak},\ a_0 = \frac{-c}{ak},\ a_1 = 0, \mu = \frac{\mp c}{2k^2}$
3. $a_{-1} = \frac{\mp c}{2ak},\ a_0 = \frac{-c}{ak}, a_1 = \frac{\mp c}{2ak}, \mu = \frac{\mp c}{4k^2}$

Where c and k are arbitrary constants. This in turn gives kink solutions:

$$u_1(x,t) = \frac{-c}{ak}\left(1 \pm \tanh\left(\frac{\mp c}{2k^2}\left(\frac{kx^\beta}{\Gamma(1+\beta)} - \frac{ct^\alpha}{\Gamma(1+\alpha)}\right)\right)\right)$$

$$u_2(x,t) = \frac{-c}{ak}\left(1 \pm \coth\left(\frac{\mp c}{2k^2}\left(\frac{kx^\beta}{\Gamma(1+\beta)} - \frac{ct^\alpha}{\Gamma(1+\alpha)}\right)\right)\right)$$

$$u_3(x,t) = \frac{-c}{2ak}\left[2 \pm \tanh\left(\frac{\mp c}{4k^2}\left(\frac{kx^\beta}{\Gamma(1+\beta)} - \frac{ct^\alpha}{\Gamma(1+\alpha)}\right)\right) \pm \coth\left(\frac{\mp c}{4k^2}\left(\frac{kx^\beta}{\Gamma(1+\beta)} - \frac{ct^\alpha}{\Gamma(1+\alpha)}\right)\right)\right]$$

2. The space-time fractional Fitzhugh Nagumo equation

$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^{2\beta} u}{\partial x^{2\beta}} - u(1-u(a-u)),\ \ 0 < \alpha, \beta < 1$(34)Substituting from Eq. (22) changes the FPDE (34) into the following nonlinear (ODE)

$cu' + k^2 u'' - u(1-u)(a-u) = 0$(35)Balancing u'' with u^3 we get $M = 1$.

Thus Eq. (24) becomes

$u(\xi) = S(Y) = a_{-1}Y^{-1} + a_0 + a_1 Y$.(36)Substituting from Eq. (25) into Eq. (35) we get

$c\mu(1-Y^2)\frac{dS}{dY} - 2Y\mu^2k^2(1-Y^2)\frac{dS}{dY} + \mu^2k^2(1-Y^2)\frac{d^2S}{dY^2} - S(1-S)(a-S) = 0$(37)Substituting Eq. (36) into Eq. (37), then by using maple package yields a system of algebraic equations for a_{-1}, a_0, a_1, and μ, c in the form:

$Y^{-3}: 2\mu^2k^2a_{-1} - a_{-1}^3 = 0$

$Y^{-2}: 3a_0a_{-1}^2 + c\mu a_{-1} - a_{-1}^2 + a_{-1}^2 a = 0$

$Y^{-1}: 2a_0a_{-1}a - 2\mu^2k^2a_{-1} + 2a_0a_{-1} - 3a_0^2a_{-1} - 3a_1a_{-1}^2 - a_{-1}a = 0$

$Y^0: 2a\_a_{-1} + c\mu a_{-1} + a_0^2 + 2a_1a_{-1}a + a_0^2a - a_0a + c\mu a_1 - 6a_0a_1a_{-1} - \quad a_0^2 = 0$

$Y: 3a_1^2a_{-1} + 3a_0^2a_1 - 2a_0a_1a - 2a_0a_1 + 2\mu^2k^2a_1 + a_1a = 0$

$Y^2: \ a_1^2 + a_1^2a - 3a_0a_1^2 - c\mu a_1 = 0$

$Y^3: 2\mu^2k^2a_1 - a_1^3 = 0$

Using Maple gives nine sets of solutions:

1.$a_{-1} = 0,\ a_0 = \frac{1}{2},\ a_1 = \frac{\pm 1}{2},\ \mu = \frac{1}{2\sqrt{2}k}, c = \frac{\mp(1-2a)k}{\sqrt{2}}$

2.$a_{-1} = 0,\ a_0 = \frac{a}{2},\quad a_1 = \frac{\pm a}{2},\quad \mu = \frac{a}{2\sqrt{2}k},\quad c = \frac{\mp(a-2)k}{\sqrt{2}}$

3.$a_{-1} = 0,\quad a_0 = \frac{a+1}{2},\ a_1 = \frac{\pm(a-1)}{2},\quad \mu = \frac{a-1}{2\sqrt{2}k},\quad c = \frac{\mp(a+1)k}{\sqrt{2}}$

4.$a_{-1} = \frac{\pm 1}{2},\ a_0 = \frac{1}{2},\ a_1 = 0,\ \mu = \frac{1}{2\sqrt{2}k}, c = \frac{\mp(1-2a)k}{\sqrt{2}}$

5.$a_{-1} = \frac{\pm a}{2},\ a_0 = \frac{a}{2},\ a_1 = 0,\quad \mu = \frac{a}{2\sqrt{2}k},\quad c = \frac{\mp(a-2)k}{\sqrt{2}}$

6. $a_{-1} = \frac{\pm(a-1)}{2},\quad a_0 = \frac{a+1}{2},\quad a_1 = 0,\quad \mu = \frac{a-1}{2\sqrt{2}k},\quad c = \frac{\mp(a+1)k}{\sqrt{2}}$

7.$a_{-1} = \frac{\pm 1}{4},\quad a_0 = \frac{1}{2},\quad a_1 = \frac{\pm 1}{4},\quad \mu = \frac{1}{4\sqrt{2}k},\quad c = \frac{\mp(1-2a)k}{\sqrt{2}}$

8.$a_{-1} = \frac{\pm a}{4},\quad a_0 = \frac{a}{2},\ a_1 = \frac{\pm a}{4},\quad \mu = \frac{a}{4\sqrt{2}k},\quad c = \frac{\mp(a-2)k}{\sqrt{2}}$

9. $a_{-1}=\frac{\pm(a-1)}{4},\ a_0=\frac{a+1}{2},\quad a_1=\frac{\pm(a-1)}{4},\ \mu=\frac{a-1}{4\sqrt{2}k},\quad c=\frac{\mp(a+1)k}{\sqrt{2}}$

Where c and k are arbitrary constants. This in turn gives kink solutions

$$u_1(x,t)=\frac{1}{2}\left(1\pm\tanh\left(\frac{1}{2\sqrt{2}k}\left(\frac{kx^\beta}{\Gamma(1+\beta)}\pm\frac{(1-2a)kt^\alpha}{\sqrt{2}\Gamma(1+\alpha)}\right)\right)\right)$$

$$u_2(x,t)=\frac{a}{2}\left(1\pm\tanh\left(\frac{a}{2\sqrt{2}k}\left(\frac{kx^\beta}{\Gamma(1+\beta)}\pm\frac{(a-2)kt^\alpha}{\sqrt{2}\Gamma(1+\alpha)}\right)\right)\right)$$

$$u_3(x,t)=\frac{a+1}{2}\pm\frac{a-1}{2}\tanh\left(\frac{a-1}{2\sqrt{2}k}\left(\frac{kx^\beta}{\Gamma(1+\beta)}\pm\frac{(a+1)kt^\alpha}{\sqrt{2}\Gamma(1+\alpha)}\right)\right)$$

$$u_4(x,t)=\frac{1}{2}\left(1\pm\coth\left(\frac{1}{2\sqrt{2}k}\left(\frac{kx^\beta}{\Gamma(1+\beta)}\pm\frac{(1-2a)kt^\alpha}{\sqrt{2}\Gamma(1+\alpha)}\right)\right)\right)$$

$$u_5(x,t)=\frac{a}{2}\left(1\pm\coth\left(\frac{a}{2\sqrt{2}k}\left(\frac{kx^\beta}{\Gamma(1+\beta)}\pm\frac{(a-2)kt^\alpha}{\sqrt{2}\Gamma(1+\alpha)}\right)\right)\right)$$

$$u_6(x,t)=\frac{a+1}{2}\pm\frac{a-1}{2}\coth\left(\frac{a-1}{2\sqrt{2}k}\left(\frac{kx^\beta}{\Gamma(1+\beta)}\pm\frac{(a+1)kt^\alpha}{\sqrt{2}\Gamma(1+\alpha)}\right)\right)$$

$$u_7(x,t)=\frac{1}{4}\left[\left(2\pm\tanh\left(\frac{1}{4\sqrt{2}k}\left(\frac{kx^\beta}{\Gamma(1+\beta)}\pm\frac{(1-2a)kt^\alpha}{\sqrt{2}\Gamma(1+\alpha)}\right)\right)\right)\right.$$
$$\left.\pm\coth\left(\frac{1}{4\sqrt{2}k}\left(\frac{kx^\beta}{\Gamma(1+\beta)}\pm\frac{(1-2a))kt^\alpha}{\sqrt{2}\Gamma(1+\alpha)}\right)\right)\right]$$

$$u_8(x,t)=\frac{a}{4}\left[(2\right.$$
$$\pm\tanh\left(\frac{a}{4\sqrt{2}k}\left(\frac{kx^\beta}{\Gamma(1+\beta)}\pm\frac{(a-2)kt^\alpha}{\sqrt{2}\Gamma(1+\alpha)}\right)\right)$$
$$\left.\pm\coth\left(\frac{a}{4\sqrt{2}k}\left(\frac{kx^\beta}{\Gamma(1+\beta)}\pm\frac{(a-2)kt^\alpha}{\sqrt{2}\Gamma(1+\alpha)}\right)\right)\right]$$

$$u_9(x,t)=\frac{a+1}{2}\pm\frac{a-1}{4}\tanh\left(\frac{a-1}{4\sqrt{2}k}\left(\frac{kx^\beta}{\Gamma(1+\beta)}\pm\frac{(a+1)kt^\alpha}{\sqrt{2}\Gamma(1+\alpha)}\right)\right)$$

$$\pm\frac{a-1}{4}\coth\left(\frac{a-1}{4\sqrt{2}k}\left(\frac{kx^{\beta}}{\Gamma(1+\beta)}\pm\frac{(a+1)kt^{\alpha}}{\sqrt{2}\Gamma(1+\alpha)}\right)\right)$$

3. The general nonlinear space-time fractional parabolic equation

$\frac{\partial^{\alpha}u}{\partial t^{\alpha}}=\frac{\partial^{2\beta}u}{\partial x^{2\beta}}+au+bu^{n},\ \ 0<\alpha,\beta<1.$(38)Substituting froe Eq. (22) changes the FPDE (38) into the following nonlinear (ODE)

$cu'+k^{2}u''+au+bu^{n}$ (39) Balancing u″ with u^n we get $M=\frac{2}{n-1}$

According the Eq. (26), we take the transformation

$u=v^{\frac{1}{n-1}}(\xi)$(40)Substituting Eq. (40) into Eq. (39) yields the (ODE)

$c(n-1)vv'+k^{2}(n-1)vv''+k^{2}(2-n)(v')^{2}+a(n-1)^{2}v^{2}+\ b(n-1)^{2}v^{3}=0$(41)With respect to v with variable ξ.Balancing$vv''$with v^{3} gives

$$M+M+2=3M$$

That gives M=2.Thus

$v(\xi)=S(Y)=a_{-2}Y^{-2}+a_{-1}Y^{-1}+a_{0}+a_{1}Y+a_{2}Y^{2}$(42) Substituting from Eq. (25) into Eq. (41) we get

$c\mu(n-1)(1-Y^{2})\frac{dS}{dY}S-2k^{2}\mu^{2}Y(1-Y^{2})\frac{dS}{dY}S+k^{2}\mu^{2}(1-Y^{2})^{2}\frac{d^{2}S}{dY^{2}}S+\ \ k^{2}(2-\ \ n)\left(\mu(1-Y^{2})\frac{dS}{dY}\right)^{2}+a(n-1)^{2}S^{2}+b(n-1)^{2}S^{3}=0$

(43*)* Substituting Eq. (42) into Eq. (43), then by using maple package we get a system of algebraic equations for a_{-2}, a_{-1}, a_0, a_1, a_2 and μ, c in the form*:*

Y^{-6}: $b n^{2} a_{-2}^{3}+b a_{-2}^{3}-2 b n a_{-2}^{3}+2 k^{2}\mu^{2} a_{-2}^{2}+2 k^{2}\mu^{2} n a_{-2}^{2}=0,$

Y^{-5}: $-2 c\mu n a_{-2}^{2}+2 c\mu a_{-2}^{2}+4 k^{2}\mu^{2} n a_{-2} a_{-1}+3 b n^{2} a_{-2}^{2} a_{-1}-6 b n a_{-2}^{2} a_{-1}+3 b a_{-2}^{2} a_{-1}=0,$

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Y^{-4}:

$$-8k^2\mu^2a_{-2}^2+an^2a_{-2}^2-6bna_0a_{-2}^2+6k^2\mu^2na_0a_{-2}$$
$$+3ba_0a_{-2}^2-2ana_{-2}^2-6bna_{-2}a_{-1}^2+3c\mu a_{-2}a_{-1}$$
$$+3bn^2a_0a_{-2}^2-3c\mu na_{-2}a_{-1}+3ba_{-2}a_{-1}^2$$
$$+3bn^2a_{-2}a_{-1}^2+aa_{-2}^2+k^2\mu^2na_{-1}^2-6k^2\mu^2a_0a_{-2}$$
$$=0,$$

Y^{-3}:

$$-2k^2\mu^2a_0a_{-1}-2c\mu a_{-2}^2-14k^2\mu^2a_{-2}a_1+2c\mu na_{-2}^2$$
$$+3ba_{-2}^2a_1+6bn^2a_{-2}a_0a_{-1}+2aa_{-2}a_{-1}+ba_{-1}^3$$
$$+3bn^2a_{-2}^2a_1+10k^2\mu^2na_{-2}a_1-2c\mu na_0a_{-2}$$
$$+2an^2a_{-2}a_{-1}+bn^2a_{-1}^3+c\mu a_{-1}^2-2k^2\mu^2na_{-2}a_{-1}$$
$$-12bna_{-2}a_0a_{-1}-4ana_{-2}a_{-1}+6ba_{-2}a_0a_{-1}$$
$$-c\mu na_{-1}^2-6k^2\mu^2a_{-2}a_{-1}+2c\mu a_0a_{-2}$$
$$+2k^2\mu^2na_0a_{-1}-6bna_{-2}^2a_1-2bna_{-1}^3=0,$$

Y^{-2}:

$$-2ana_{-1}^2+3ba_{-1}^2a_0+2aa_0a_{-2}+an^2a_{-1}^2-2k^2\mu^2a_{-1}^2$$
$$-c\mu na_0a_{-1}-c\mu na_{-2}a_1+3c\mu na_{-2}a_{-1}$$
$$+6bn^2a_{-2}a_{-1}a_1-12bna_{-2}a_{-1}a_1+4k^2\mu^2na_{-1}a_1$$
$$-8k^2\mu^2na_0a_{-2}+16k^2\mu^2na_{-2}a_2+3ba_{-2}^2a_2$$
$$+3ba_{-2}a_0^2+6k^2\mu^2a_{-2}^2-2k^2\mu^2na_{-2}^2+3bn^2a_{-2}$$
$$a_0^2+3bn^2a_{-2}^2a_2+8k^2\mu^2a_0a_{-2}-24k^2\mu^2a_{-2}a_2$$
$$-4ana_0a_{-2}+2an^2a_0a_{-2}-6k^2\mu^2a_{-1}a_1+3bn^2$$
$$a_{-1}^2a_0-6bna_{-2}a_0^2+c\mu a_{-2}a_1+c\mu a_0a_{-1}$$
$$-3c\mu a_{-2}a_{-1}+6ba_{-2}a_{-1}a_1-6bna_{-1}^2a_0-6bn$$
$$a_{-2}^2a_2+aa_{-1}^2=0,$$

Y^{-1}:

$$
\begin{aligned}
&2aa_0a_{-1} + 3ba_{-1}a_0^2 - c\mu a_{-1}^2 + 2aa_{-2}a_1 + 3ba_{-1}^2a_1 \\
&+ 2c\mu n a_0 a_{-2} + 6bn^2 a_{-2}a_{-1}a_2 + 6bn^2 a_{-2}a_0a_1 \\
&- 12bna_{-2}a_{-1}a_2 - 12bna_{-2}a_0a_1 \\
&- 2k^2\mu^2 n a_{-2}a_{-1} - 18k^2\mu^2 n a_{-2}a_1 + 8k^2\mu^2 n a_{-1}a_2 \\
&- 2k^2\mu^2 n a_0 a_{-1} + 3bn^2 a_{-1}^2 a_1 + 3bn^2 a_{-1}a_0^2 \\
&+ 2an^2 a_0 a_{-1} + 2an^2 a_{-2}a_1 - 12k^2\mu^2 a_{-1}a_2 \\
&- 4ana_0a_{-1} + 6k^2\mu^2 a_{-2}a_{-1} + 6ba_{-2}a_0a_1 \\
&+ 26k^2\mu^2 a_{-2}a_1 - 4ana_{-2}a_1 + 2k^2\mu^2 a_0a_{-1} + c\mu n \\
&a_{-1}^2 - 2c\mu a_0 a_{-2} + 6ba_{-2}a_{-1}a_2 - 6bna_{-1}^2a_1 \\
&- 6bna_{-1}a_0^2 = 0,
\end{aligned}
$$

Y^0:

$$
\begin{aligned}
&3ba_{-1}^2a_2 + an^2a_0^2 + 2k^2\mu^2 a_1^2 + 2aa_{-1}a_1 + 2aa_{-2}a_2 \\
&- 2ana_0^2 + 2k^2\mu^2 a_{-1}^2 + c\mu n a_0 a_{-1} + c\mu n a_{-2}a_1 \\
&- 12bna_{-2}a_0a_2 + 6bn^2 a_{-2}a_0a_2 + 6bn^2 a_{-1}a_0a_1 \\
&- 12bna_{-1}a_0a_1 - 8k^2\mu^2 n a_{-1}a_1 + 2k^2\mu^2 n a_0 a_{-2} \\
&+ 2k^2\mu^2 n a_0 a_2 - 32k^2\mu^2 n a_{-2}a_2 + bn^2 a_0^3 + 3ba_{-2} \\
&a_1^2 - 2bna_0^3 + 3bn^2 a_{-1}^2 a_2 - 4ana_{-1}a_1 \\
&- 4ana_{-2}a_2 + 2an^2 a_{-2}a_2 - 2k^2\mu^2 a_0 a_{-2} \\
&+ 6ba_{-1}a_0a_1 + 3bn^2 a_{-2}a_1^2 + 6ba_{-2}a_0a_2 \\
&+ 48k^2\mu^2 a_{-2}a_2 - 2k^2\mu^2 a_0a_2 + 2an^2 a_{-1}a_1 \\
&+ 12k^2\mu^2 a_{-1}a_1 - 6bna_{-2}a_1^2 - c\mu a_{-2}a_1 \\
&- c\mu a_0 a_{-1} - k^2\mu^2 n a_{-1}^2 - c\mu a_{-1}a_2 - c\mu a_0 a_1 \\
&- k^2\mu^2 n a_1^2 - 6bna_{-1}^2 a_2 + aa_0^2 + c\mu n a_{-1}a_2 \\
&+ c\mu n a_0 a_1 + ba_0^3 = 0,
\end{aligned}
$$

Y:

$$
\begin{aligned}
&3ba_0^2a_1+2aa_0a_1-c\mu a_1^2+3ba_{-1}a_1^2+2aa_{-1}a_2\\
&+2c\mu na_0a_2-2k^2\mu^2na_0a_1-2k^2\mu^2na_1a_2\\
&+6bn^2a_{-2}a_1a_2+6bn^2a_{-1}a_0a_2-12bna_{-2}a_1a_2\\
&-12bna_{-1}a_0a_2+8k^2\mu^2na_{-2}a_1-18k^2\mu^2na_{-1}a_2\\
&+3bn^2a_{-1}a_1^2-2c\mu a_0a_2-6bna_0^2a_1\\
&+6ba_{-1}a_0a_2-4ana_0a_1+2k^2\mu^2a_0a_1\\
&+6k^2\mu^2a_1a_2+2an^2a_{-1}a_2-4ana_{-1}a_2\\
&+26k^2\mu^2a_{-1}a_2-12k^2\mu^2a_{-2}a_1+2an^2a_0a_1\\
&+6ba_{-2}a_1a_2+3bn^2a_0^2a_1+c\mu na_1^2-6bna_{-1}a_1^2\\
&=0,
\end{aligned}
$$

Y²:

$$
\begin{aligned}
&-2ana_1^2+an^2a_1^2+2aa_0a_2-2k^2\mu^2a_1^2+3ba_0^2a_2\\
&+3ba_0a_1^2+3c\mu na_1a_2+6bn^2a_{-1}a_1a_2\\
&-12bna_{-1}a_1a_2+4k^2\mu^2na_{-1}a_1-8k^2\mu^2na_0a_2\\
&+16k^2\mu^2na_{-2}a_2+3ba_{-2}a_2^2+6k^2\mu^2a_2^2\\
&+3bn^2a_{-2}a_2^2-2k^2\mu^2na_2^2-4ana_0a_2+2an^2a_0a_2\\
&-24k^2\mu^2a_{-2}a_2+8k^2\mu^2a_0a_2-6bna_0a_1^2\\
&-3c\mu a_1a_2-6bna_0^2a_2-6k^2\mu^2a_{-1}a_1\\
&+6ba_{-1}a_1a_2+3bn^2a_0^2a_2+3bn^2a_0a_1^2-6bna_{-2}\\
&a_2^2+c\mu a_{-1}a_2+c\mu a_0a_1+aa_1^2-c\mu na_{-1}a_2\\
&-c\mu na_0a_1=0,
\end{aligned}
$$

Y^3:

$$-2k^2\mu^2 a_0 a_1 - 6k^2\mu^2 a_1 a_2 + 10k^2\mu^2 n a_{-1} a_2 - 6bna_{-1} a_2^2 - 2bna_1^3 + 3bn^2 a_{-1} a_2^2 + 6bn^2 a_0 a_1 a_2 + 2c\mu n a_2^2 - 12bna_0 a_1 a_2 - 14k^2\mu^2 a_{-1} a_2 - 2k^2\mu^2 n a_1 a_2 - 4ana_1 a_2 + 2aa_1 a_2 + 2k^2\mu^2 n a_0 a_1 + 2an^2 a_1 a_2 + 3ba_{-1} a_2^2 + ba_1^3 + 2c\mu a_0 a_2 + c\mu a_1^2 + bn^2 a_1^3 + 6ba_0 a_1 a_2 - c\mu n a_1^2 - 2c\mu a_2^2 - 2c\mu n a_0 a_2 = 0,$$

Y^4:

$$6bna_1^2 a_2 - 6k^2\mu^2 a_0 a_2 + 6k^2\mu^2 n a_0 a_2 - 3c\mu n a_1 a_2 - 6bna_0 a_2^2 + k^2\mu^2 n a_1^2 + an^2 a_2^2 + 3ba_1^2 a_2 + 3bn^2 a_1^2 a_2 + aa_2^2 + 3bn^2 a_0 a_2^2 + 3ba_0 a_2^2 - 8k^2\mu^2 a_2^2 - 2ana_2^2 + 3c\mu a_1 a_2 = 0,$$

Y^5: $$4k^2\mu^2 n a_1 a_2 + 2c\mu a_2^2 + 3ba_1 a_2^2 + 3bn^2 a_1 a_2^2 - 2c\mu n a_2^2 - 6bna_1 a_2^2 = 0,$$

Y^6: $$-2bna_2^3 + ba_2^3 + 2k^2\mu^2 n a_2^2 + 2k^2\mu^2 a_2^2 + bn^2 a_2^3 = 0.$$

Maple gives three sets of solutions:

1. $a_{-2} = 0, a_{-1} = 0,\ \ a_0 = \frac{-a}{4b}, a_1 = \frac{\mp a}{2b}, a_2 = \frac{-a}{4b}, c = \mp(n+3)\sqrt{\frac{a}{2(n+1)}}k, \mu = \frac{(n-1)}{2k}\sqrt{\frac{a}{2(n+1)}}$, n>1, a>0

2. $a_{-2} = \frac{-a}{4b}, a_{-1} = \frac{\mp a}{2b} a_0 = \frac{-a}{4b}, a_1 = 0, a_2 = 0, c = \mp(n+3)\sqrt{\frac{a}{2(n+1)}}k, \mu = \frac{(n-1)}{2k}\sqrt{\frac{a}{2(n+1)}}$,

n>1, a>0

3. $a_{-2} = \frac{-a}{16b}, a_{-1} = \frac{\mp a}{4b}, a_0 = \frac{-3a}{8b}, a_1 = \frac{\mp a}{4b}, a_2 \frac{-a}{16b},\ c = \mp(n+\ 3)\sqrt{\frac{a}{2(n+1)}}k, \mu = \frac{(n-1)}{4k}\sqrt{\frac{a}{2(n+1)}},$

n>1, a>0

This in turn gives the solutions as follows*:*

If a>0 we obtain the kink solutions

$$u_1(x,t) = \left\{\frac{-a}{4b}\left(1 \pm \tanh\left(\frac{(n-1)}{2k}\sqrt{\frac{a}{2(n+1)}}\left(\frac{kx^{\beta}}{\Gamma(1+\beta)} \pm \frac{(n+3)\sqrt{\frac{a}{2(n+1)}}kt^{\alpha}}{\Gamma(1+\alpha)}\right)\right)\right)^2\right\}^{\frac{1}{n-1}}$$

$$u_2(x,t) = \left\{\frac{-a}{4b}\left(1 \pm \coth\left(\frac{(n-1)}{2k}\sqrt{\frac{a}{2(n+1)}}\left(\frac{kx^{\beta}}{\Gamma(1+\beta)} \pm \frac{(n+3)\sqrt{\frac{a}{2(n+1)}}kt^{\alpha}}{\Gamma(1+\alpha)}\right)\right)\right)^2\right\}^{\frac{1}{n-1}}$$

$$u_3(x,t) = \left\{\frac{a}{16b}\left[2 - \left(2 \pm \tanh\left(\frac{(n-1)}{4k}\sqrt{\frac{a}{2(n+1)}}\left(\frac{kx^{\beta}}{\Gamma(1+\beta)} \pm \frac{(n+3)\sqrt{\frac{a}{2(n+1)}}kt^{\alpha}}{\Gamma(1+\alpha)}\right)\right)\right)^2 - \left(2 \pm \coth\left(\frac{(n-1)}{4k}\sqrt{\frac{a}{2(n+1)}}\left(\frac{kx^{\beta}}{\Gamma(1+\beta)} \pm \frac{(n+3)\sqrt{\frac{a}{2(n+1)}}kt^{\alpha}}{\Gamma(1+\alpha)}\right)\right)\right)^2\right]\right\}^{\frac{1}{n-1}}$$

.

If a<0, the first tow solutions give the periodic solutions:

$$u_1(x,t) = \left\{\frac{-a}{4b}\left(1 \pm \tan^2\left(\frac{(n-1)}{2k}\sqrt{\frac{-a}{2(n+1)}}\left(\frac{kx^{\beta}}{\Gamma(1+\beta)} \pm \frac{(n+3)\sqrt{\frac{a}{2(n+1)}}kt^{\alpha}}{\Gamma(1+\alpha)}\right)\right)\right)\right\}^{\frac{1}{n-1}}$$

$$u_2(x,t) = \left\{\frac{-a}{4b}\left(1 \pm \cot^2\left(\frac{(n-1)}{2k}\sqrt{\frac{-a}{2(n+1)}}\left(\frac{kx^{\beta}}{\Gamma(1+\beta)} \pm \frac{(n+3)\sqrt{\frac{a}{2(n+1)}}kt^{\alpha}}{\Gamma(1+\alpha)}\right)\right)\right)\right\}^{\frac{1}{n-1}}$$

And the thirdsolution gives a complex solution:

$$u_3(x,t) = \left\{\frac{-a}{16b}\left[\left(1 \pm i\tanh\left(\frac{(n-1)}{4k}\sqrt{\frac{-a}{2(n+1)}}\left(\frac{kx^{\beta}}{\Gamma(1+\beta)} \pm \frac{(n+3)\sqrt{\frac{a}{2(n+1)}}kt^{\alpha}}{\Gamma(1+\alpha)}\right)\right)\right)\right.\right.$$

$$\left(3 \pm i\tanh\left(\frac{(n-1)}{4k}\sqrt{\frac{-a}{2(n+1)}}\left(\frac{kx^{\beta}}{\Gamma(1+\beta)} \pm \frac{(n+3)\sqrt{\frac{a}{2(n+1)}}kt^{\alpha}}{\Gamma(1+\alpha)}\right)\right)\right)$$

$$+\left(\pm i\coth\left(\frac{(n-1)}{2k}\sqrt{\frac{-a}{2(n+1)}}\left(\frac{kx^{\beta}}{\Gamma(1+\beta)} \pm \frac{(n+3)\sqrt{\frac{a}{2(n+1)}}kt^{\alpha}}{\Gamma(1+\alpha)}\right)\right)\right)$$

$$\left.\left.\left(3 \pm i\coth\left(\frac{(n-1)}{4k}\sqrt{\frac{-a}{2(n+1)}}\left(\frac{kx^{\beta}}{\Gamma(1+\beta)} \pm \frac{(n+3)\sqrt{\frac{a}{2(n+1)}}kt^{\alpha}}{\Gamma(1+\alpha)}\right)\right)\right)\right]\right\}^{\frac{1}{n-1}}$$

5. CONCLUSIONS

It is clear that if we set α=β=1 in the solutions that we have obtained by using Tanh-coth method, and with the aid of the Maple, then we get solutions contained the solutions obtained by Wazwaz [1]. (Comp. [24- 28]).

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